Universidad Católica del Norte, Economic Department

Spatial Econometrics: Final Exam

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December 13, 2015

1. Consider the following SAC model with heteroskedastic errors:

$$\mathbf{y} = \rho \mathbf{W}_1 \mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \quad (1)$$
$$\mathbf{u} = \lambda \mathbf{W}_2 \mathbf{u} + \boldsymbol{\varepsilon} \quad (2)$$
$$\boldsymbol{\varepsilon} \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Omega}) \quad (3)$$

The matrix $\boldsymbol{\Omega}$ is the variance-covariance matrix of the error terms, which is assummed to be known a priori. For example, we can assume that:

$$\mathrm{Var}(\epsilon_i) = \sigma_i^2 = \mathbf{z}_i^\top \boldsymbol{\alpha} \quad (4)$$

or

$$\mathrm{Var}(\epsilon_i) = \sigma_i^2 = \exp(\mathbf{z}_i^\top \boldsymbol{\alpha}) \quad (5)$$

or more general,

$$\mathrm{Var}(\epsilon_i) = \sigma_i^2 = \mathbf{h}(\mathbf{z}_i^\top \boldsymbol{\alpha}) \quad (6)$$

where $\mathbf{h}(\cdot)$ is any function, $\mathbf{z}_i$ is a vector of covariates for each spatial unit, and $\boldsymbol{\alpha}$ is a vector of parameters with element $\alpha_p, p = 0, 1, ..., P$. Therefore, the diagonal elements of the error covariance matrix $\boldsymbol{\Omega}$ are:

$$\boldsymbol{\Omega}_{ii} = \mathbf{h}_i(\mathbf{z}_i^\top \boldsymbol{\alpha}), \quad \mathbf{h}_i > 0 \quad (7)$$

Note that the model has $2 + K + P$ unknown parameters:

$$\boldsymbol{\theta} = (\rho, \boldsymbol{\beta}^\top, \lambda, \boldsymbol{\alpha}^\top)^\top. \tag{8}$$

a) Find the Log-likelihood function. Show every step.

b) Find the first order conditions. Show all your work.

2. Explain and derive the LR test for the SLM model. Show all your work.

3. Explain how to estimate a SEM model using the Method of Moments and the Spatial FGLS. Show all your work.

Some important results are the followings:

$$\frac{\partial(\rho\mathbf{W})}{\partial\rho} = \mathbf{W} \tag{9}$$

$$\begin{aligned}\frac{\partial\mathbf{A}}{\partial\rho} &= \frac{\partial(\mathbf{I} - \rho\mathbf{W})}{\partial\rho} \\ &= \frac{\partial\mathbf{I}}{\partial\rho} - \frac{\partial\rho\mathbf{W}}{\rho} \\ &= -\mathbf{W}\end{aligned} \tag{10}$$

$$\frac{\partial\log|\mathbf{A}|}{\partial\rho} = \operatorname{tr}(\mathbf{A}^{-1}\partial\mathbf{A}/\partial\rho) = \operatorname{tr}\left[\mathbf{A}^{-1}(-\mathbf{W})\right] \tag{11}$$

Let $\boldsymbol{\varepsilon} = \mathbf{Ay} - \mathbf{X}\boldsymbol{\beta}$, then:

$$\frac{\partial\varepsilon}{\partial\rho} = \frac{\partial(\mathbf{Ay} - \mathbf{X}\boldsymbol{\beta})}{\partial\rho} = -\mathbf{Wy} \tag{12}$$

$$\frac{\partial\boldsymbol{\varepsilon}^\top\boldsymbol{\varepsilon}}{\partial\rho} = \boldsymbol{\varepsilon}^\top(\partial\varepsilon/\partial\rho) + (\partial\varepsilon^\top/\partial\rho)\boldsymbol{\varepsilon} = 2\boldsymbol{\varepsilon}^\top(\partial\varepsilon/\partial\rho) = 2\boldsymbol{\varepsilon}^\top(-\mathbf{W})\mathbf{y} \tag{13}$$

$$\frac{\partial\mathbf{A}^{-1}}{\partial\rho} = -\mathbf{A}^{-1}(\partial\mathbf{A}/\partial\rho)\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{W}\mathbf{A}^{-1} \tag{14}$$

$$\frac{\partial\operatorname{tr}\left(\mathbf{A}^{-1}\mathbf{W}\right)}{\partial\rho} = \operatorname{tr}\left(\partial\mathbf{A}^{-1}\mathbf{W}/\partial\rho\right) \tag{15}$$