Universidad Católica del Norte, Economic Department

Spatial Econometrics: Exam 1

Professor: Mauricio Sarrias

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- If you believe a question is unclear, please state how you interpret the question.
- Please, use formal mathematical language wherever possible.
- You must show all work for partial credit to be awarded.
- Total points: 38.

1 Questions

1. (Omitted Variables) Consider the following "deterministic" model:

$$\mathbf{y} = \mathbf{x}\beta + \mathbf{z}\theta$$

For simplicity assume that the $n \times 1$ vectors $\mathbf{x}$ and $\mathbf{z}$ are distributed $\mathrm{N}(0, \mathbf{I}_n)$.

a) (2 pts) Assume that $\mathbf{z}$ is not observed (because you do not have data), therefore the model is $\mathbf{y} = \mathbf{x}\beta + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} = \mathbf{z}\theta$. Assume that $\mathbf{z}$ is uncorrelated with the observed variable $\mathbf{x}$. Is $\widehat{\beta}_{OLS}$ biased?.

b) Assume that z exhibits zero covariance with the vector x, but now we have the following spatial autoregressive process:

$$\mathbf{z} = \rho\mathbf{W}\mathbf{z} + \mathbf{r}$$
$$\mathbf{z} = (\mathbf{I}_n - \rho\mathbf{W})^{-1}\mathbf{r}$$

where $\mathbf{r}$ is a $n \times 1$ vector os disturbances distributed $\mathrm{N}(\mathbf{0}, \sigma^2_r\mathbf{I}_n)$, and $\mathbf{W}$ is the usual spatial weight matrix.

i. (1 pts) Please, give an example of a model with the characteristics above.

ii. (1 pts) Derive the reduced form equation by setting $\theta\mathbf{r} = \mathbf{u}$

iii. (1 pts) Find $\mathbb{E}(\mathbf{y}|\mathbf{W}, \mathbf{z}, \mathbf{x})$.

iv. (2 pts) Assume that $\mathbf{u} = \mathbf{x}\gamma + \boldsymbol{v}$, where $\boldsymbol{v}$ is independent of $\mathbf{x}$ and $\boldsymbol{v} \sim \mathrm{N}(\mathbf{0}, \sigma_v^2 \mathbf{I}_n)$. Find the reduced equation.

v. (2 pts) Considering your answer from (iv), is $\widehat{\beta}$ unbiased?

vi. (3 pts) Show that the model you found in (iv) is the SDM model.

2. (3pts) (Asymptotics) Explain the two types of asymptotic in the context of spatial observations.

3. (ML estimation) Consider the following SAC model:

$$\begin{aligned}
\mathbf{y} &= \rho\mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \\
\mathbf{u} &= \lambda\mathbf{M}\mathbf{u} + \boldsymbol{\varepsilon} \\
\boldsymbol{\varepsilon} &\sim \mathrm{N}(\mathbf{0}, \sigma^2\mathbf{I}_n)
\end{aligned}$$

a) (5 pts) Find the log-likelihood function (Show very step).

b) (10 pts) Find the first order conditions (Show all your work).

c) (8 pts) Derive the Likelihood Ratio test for $H_0 : \rho = 0$.

Some important results are the followings:

$$\frac{\partial(\rho\mathbf{W})}{\partial\rho} = \mathbf{W} \tag{1}$$

$$\begin{aligned}
\frac{\partial\mathbf{A}}{\partial\rho} &= \frac{\partial(\mathbf{I} - \rho\mathbf{W})}{\partial\rho} \\
&= \frac{\partial\mathbf{I}}{\partial\rho} - \frac{\partial\rho\mathbf{W}}{\rho} \\
&= -\mathbf{W}
\end{aligned} \tag{2}$$

$$\frac{\partial\log|\mathbf{A}|}{\partial\rho} = \mathrm{tr}(\mathbf{A}^{-1}\partial\mathbf{A}/\partial\rho) = \mathrm{tr}\left[\mathbf{A}^{-1}(-\mathbf{W})\right] \tag{3}$$

Let $\boldsymbol{\varepsilon} = \mathbf{A}\mathbf{y} - \mathbf{X}\boldsymbol{\beta}$, then:

$$\frac{\partial\varepsilon}{\partial\rho} = \frac{\partial(\mathbf{A}\mathbf{y} - \mathbf{X}\boldsymbol{\beta})}{\partial\rho} = -\mathbf{W}\mathbf{y} \tag{4}$$

$$\frac{\partial\boldsymbol{\varepsilon}^\top\boldsymbol{\varepsilon}}{\partial\rho} = \boldsymbol{\varepsilon}^\top(\partial\varepsilon/\partial\rho) + (\partial\boldsymbol{\varepsilon}^\top/\partial\rho)\boldsymbol{\varepsilon} = 2\boldsymbol{\varepsilon}^\top(\partial\varepsilon/\partial\rho) = 2\boldsymbol{\varepsilon}^\top(-\mathbf{W})\mathbf{y} \tag{5}$$

$$\frac{\partial\mathbf{A}^{-1}}{\partial\rho} = -\mathbf{A}^{-1}(\partial\mathbf{A}/\partial\rho)\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{W}\mathbf{A}^{-1} \tag{6}$$

$$\frac{\partial \operatorname{tr}\left(\mathbf{A}^{-1}\mathbf{W}\right)}{\partial \rho} = \operatorname{tr}\left(\partial \mathbf{A}^{-1}\mathbf{W}/\partial \rho\right) \tag{7}$$